Unit 9: Definite Integration

9.1 Definite integral

The definite integral can be defined as the limit of a sum.

The relation $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$ should be introduced in an intuitive approach.

Students should be reminded not to confuse a definite integral with an indefinite integral. The definite integral $\int_a^b f(x)\,\mathrm{d}x$ is a real number; whereas the indefinite integral $\int f(x)\,\mathrm{d}x$ denotes the anti-derivative of $f(x)$.

<u>Example 1</u>

A company manufactures x T.V. sets per month. The marginal profit of the monthly profit (in dollars) $P(x)$ is given by $\frac{\mathrm{d}P(x)}{\mathrm{d}x} = 165 - 0.1x$, $0 \le x \le 4000$.

The company is currently manufacturing 1600 sets per month, but is planning to increase production. Find the change in the monthly profit if the monthly production is increased from 1600 sets to 2000 sets.

9.2 Properties of definite integral

Sufficient examples should be given to illustrate the following properties of definite integrals.

(i) $\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$

(ii) $\int_a^b [f(x) \pm g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$

(iii) $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

(iv) $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$

(v) $\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(u)\,\mathrm{d}u$

(vi) $\int_a^a f(x)\,\mathrm{d}x = 0$

Students should be reminded that when using the method of substitution to perform integration, the upper and lower limits of the definite integral should be changed accordingly.

9.3 Plane areas and Applications

The close relation between definite integral and the area under a curve should be observed. However, negative area and the negative value of a definite integral are different. Various situations should be illustrated with examples to students.

Area between the curve $y = f(x)$ and the x-axis, area between the curve $x = g(y)$ and the y-axis and area between two curves should be a simple application of previous knowledge. Sketches of curves under consideration could be provided to help students' thinking.

Students should be encouraged to get into the habit of drawing sketches of the functions involved in order to have a picture of the areas of interest. They could then easily identify

pertinent boundaries and realize how to find the areas concerned.

Example 1

An electric company has proposed building a nuclear power plant on the outskirts of a major metropolitan area. In order to persuade the public that the power plant is safe enough in case an accident occurs, the company estimates that the rate at which deaths would occur within the metropolitan area because of radioactive fallout is given by the function $r(t) = 20000e^{-0.2t}$ where $r(t)$ represents the rate of deaths in persons per day and t (measured in days) represents time elapsed after the accident occurs.

(i) Find the total number of deaths 2 days after an accident occurs.
(ii) Find the total number of deaths from 2nd day to 7th day after an accident occurs.
(iii) Find the total number of deaths 7 days after an accident occurs.
(iv) How long would it take for all people in the metropolitan area to succumb to the effects of the radioactivity?

Example 2

A company specialising in a mail-order sales approach is beginning a promotional campaign. Advertising expenditure will cost the firm \$5,950 per day. Marketing specialists estimate that the rate at which profit exclusive of advertising costs generated from promotion sales decreases over promotion period. Specifically, the profit rate $f(t)$ in dollars per day at t th day after the campaign has begun is $f(t) = -50t^2 + 10000$. The company designs to conduct the campaign as long as $f(t)$ exceeds the daily advertising cost.

(i) How long should the campaign be conducted?
(ii) Find the total advertising expenditures of the campaign during that period.
(iii) Find the total profit exclusive of advertising cost for the campaign during that period.
(iv) What net profit will be expected? .
(v) Assuming that expenditures will be incurred at a rate of \$5,000 per day and that $f(t) = -10t^2 + 9000$, what will be the net profit?

9.4 Approximation of definite integrals using the trapezoidal rule

Some indefinite integrals are not readily integrable (e.g. $\int_1^2 e^{x^2}\,\mathrm{d}x$). Approximate methods have to be used to find out the answer. Application of the trapezoidal rule on a definite integral can always give an approximation.

Formula of the trapezoidal rule should be derived and illustrated with some examples.

It should be pointed out that if a curve is convex upward (e.g. $y = -x^2 + 4$), the trapezoidal rule under-estimates the required area. The reverse is the case if the curve is concave upwards (e.g. $y = x^2 + 4$). Error estimation is not required.

Better approximations can be obtained by increasing the number of subintervals.

Example 1

Use the trapezoidal rule with 2 ordinates, 4 ordinates & 8 ordinates, to estimate $\int_0^2 e^x \, dx$, correct the answer to 4 significant figures. Which is the better approximation?

Example 2

The speed of a train is shown by the following table:

t (h)	0	0.1	0.2	0.3	0.4
Speed (km/h)	10	15	23	42	58

Draw a sketch of the speed-time graph of the train. Use the trapezoidal rule to find the total distance travelled by the train between $t = 0$ and $t = 0.4$.

Example 3

Let $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$ where $0 \le x \le 1$ and $I = \int_0^1 f(x) \, dx$.

(i) Find the estimate I_1 of I using the trapezoidal rule with 5 sub intervals.

(ii) Find $f'(x)$ and $f''(x)$.

(iii) Using (ii) or otherwise, state whether I_1 in (i) is an over-estimate or under-estimate of I. Explain your answer briefly.