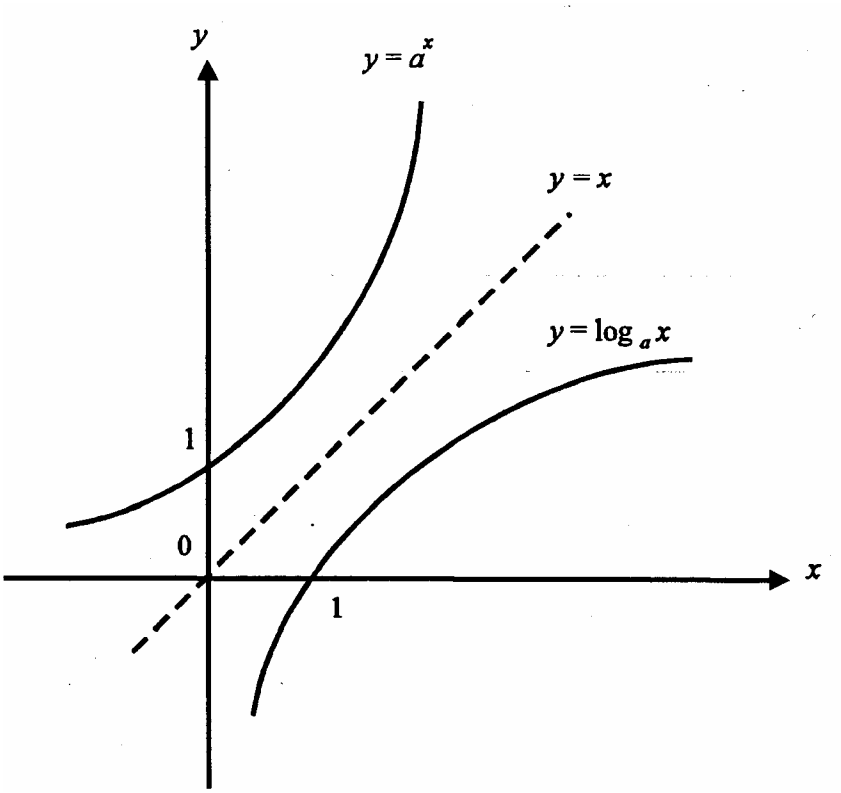
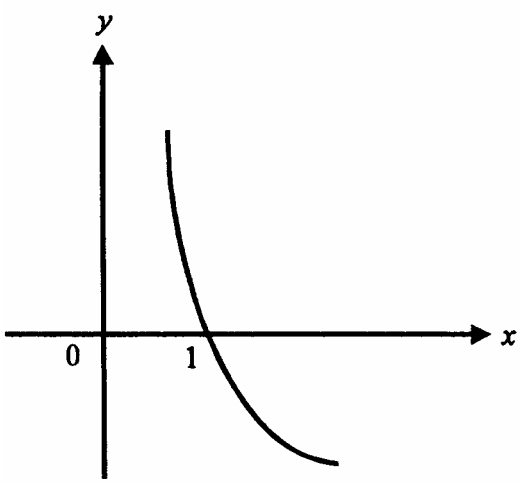
Unit 4: Logarithmic Functions

4.1 Properties and graphs of logarithmic functions $f(x) = \log_a x$ for $a > 0$, $a \neq 1$

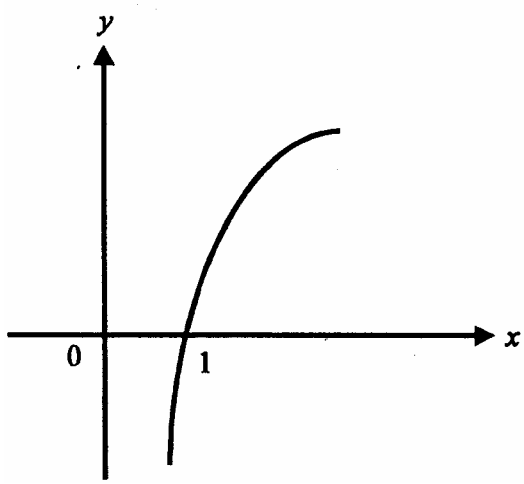
A Logarithmic function is the inverse function of the corresponding exponential function. The following diagram shows the graphs of $y = x$, $y = a^x$ and $y = \log_a x$. It can be observed that the graph of $y = \log_a x$ is the reflection in the line $y = x$ of the graph of $y = a^x$.



The following properties of logarithmic functions can be read from the graphs of $y = \log_a x$.



Graph of
$y = \log_a x$ when $0 < a < 1$



Graph of
$y = \log_a x$ when $a > 1$

(i) $\log_a x$ is defined for $x > 0$.

(ii) For $0 < a < 1$, $\log_a x$ decreases when x increases and $\log_a x \begin{cases} > 0 & \text{for all } x < 1, \\ = 0 & \text{for all } x = 1, \\ < 0 & \text{for all } x > 1. \end{cases}$

(iii) For $a > 1$, $\log_a x$ increases when x increases and $\log_a x \begin{cases} < 0 & \text{for all } x < 1, \\ = 0 & \text{for all } x = 1, \\ > 0 & \text{for all } x > 1. \end{cases}$

(iv) $\log_a a = 1$.

The laws of logarithms can be summarised as follows:

(a) $\log_a(xy) = \log_a x + \log_a y$

(b) $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

(c) $\log_a x^n = n\log_a x$

The proofs of the above could be provided, however students are not expected to know the change of base.

<u>Example 1</u>
The magnitude of an earthquake is defined in terms of the Richter Scale by $R = \log_{10}(I)$ where
R is the scale number and I is the relative intensity of the shock.
(i) Rewrite this rule in exponential form.
(ii) For the Indonesian earthquake in June 1997 the Richter Scale number was 6.5 and for the Japanese earthquake in May 1997 the Richter Scale number was 7. How many times more intense was the Japanese earthquake?

In particular case, when base $a = e$, the notation $\log_e x$ or $\ln x$ represents the natural logarithm which is the inverse function of the exponential function. Natural logarithm should be discussed and its importance mentioned.

<u>Example 2</u>
The following data relate to the cooling of a steel bar:

time t (h)	0.5	1	1.5	2	2.5	3
temperature T ($°C$)	185	137	100	75	56	42.5

Show that the relationship between temperature and time is given by $T = T_0 e^{-kt}$, where T_0 and k are constants. When will the bar cool down to 50 $°C$?

4.2 Solution of simple equations involving logarithms

Students are expected to solve the equations of the following kind:

(i) $\log_4(x-2)+\log_4(x+1)=1$

(ii) $3\cdot 2^x = 5^{x-1}$

Note: Checking of roots is essential.

4.3 Reduction of the relations $y = kx^n$ and $y = ka^x$ to a linear relations

If $y = kx^n$, then taking logarithms of both sides yields $Y = nX + c$, where $X = \log_{10} x$, $Y = \log_{10} y$ and $c = \log_{10} k$.

The equation $Y = nX + c$ represents a straight line on the X–Y co-ordinate system with slope n and Y-intercept c.

Usually, the values of x and y are found experimentally. Thus, if pairs of x, y values are given, students can plot the graph of $\log_{10} y$ against $\log_{10} x$ from which they can read off the <u>slope</u> and <u>Y-intercept</u> to determine the values of n and k.

Similarly, if $y = ka^x$, by plotting the graph of $\log_{10} y$ against x, the values of a and k can be determined.

<u>Example 1</u>
The illumination of a light bulb for various values of the distance from the light bulb is given in the following table:

Illumination I (candela)	500	31.25	17.58	7.8	6.17
Distance d (metres)	0.3	1.2	1.6	2.4	2.7

Find a law of the form $y = kd^n$.