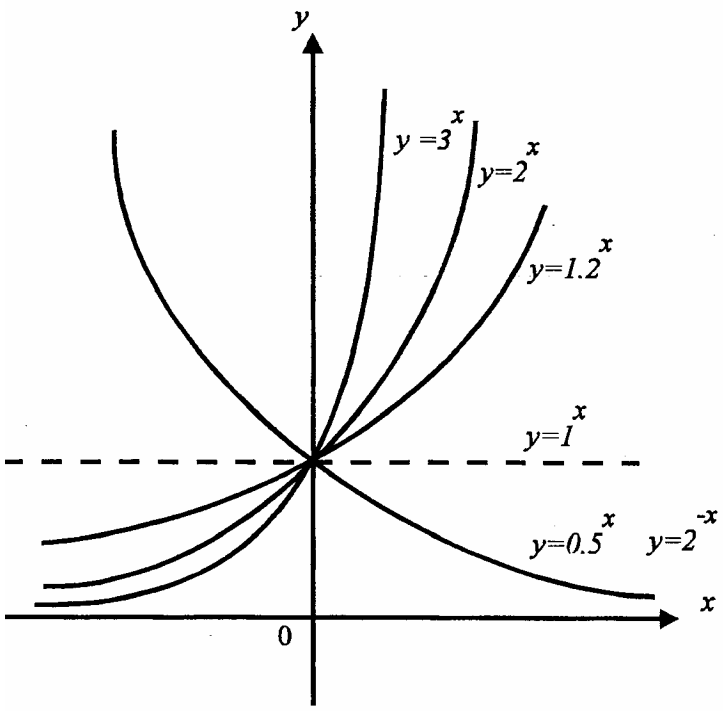
UNIT 3: Exponential Functions

3.1 Properties and graphs of exponential functions $f(x) = a^x$ for $a > 0$, $a \neq 1$

Students may be asked to plot the graph of, say $y = 2^x$ for $-3 \le x \le 5$. This function, in which the index is variable, is called an exponential function. Students should distinguish between the shapes of the graph of $f(x) = x^2$ and $f(x) = 2^x$.

In the following diagram, the graphs of $y = a^x$ for various positive values of a are shown.



Through this diagram, the students will have some idea about the general shapes of the graphs of (i) $y = a^x$ for $a > 1$ and (ii) $y = a^x$ for $0 < a < 1$. They can see that each graph lies entirely above the x-axis and cuts the y-axis at the point (0, 1).

In particular, when $a = e$ where $e = 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \cdots + \frac{1^r}{r!} + \cdots \approx 2.71828$,

$y = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$ is called the exponential series.

The laws of exponents should be revised:

(i) $a^x > 0$ for all real values of x

(ii) $a^p \cdot a^q = a^{p+q}$

(iii) $a^p \div a^q = a^{p-q}$

(iv) $(a^p)^q = a^{pq}$ and

(v) $a^0 = 1$

Many of the natural laws of the universe are of an exponential type, for example population growth, growth of bacteria, the cooling rate of substances, the loss of thermal energy of a body, growth and decay of matter and statistical functions. Mathematically they are represented as $y = a^x$.

Several important probability distribution function is calculated from the differentiation and integration of some exponential functions. This part will be further discussed in later units.

The growth of an exponential function $y = a^x$ for $a > 1$ resembles approximately that of investment at compound interest at constant rate:

$$A(n) = P\left(1 + \frac{r}{100}\right)^n$$

The growth may be termed Compound Interest Law(C.I.L.). Exponential functions are very important in science and technology since many phenomena follow the C.I.L..

There is the law of decay

$$y = ak^{-bx}$$

It describes the fact that a quantity decreases at a rate which is constantly proportional to its magnitude at any given instant. A good example is the exponential decay of radioactive element:

$$N = N_0 e^{-\lambda t}$$

Teachers may touch upon $\frac{\mathrm{d}N}{\mathrm{d}t}$, which provides good illustration on the properties of exponential functions, after they have completed differentiation.

3.2 Solution of simple equations with unknown indices

Students are expected to know how to solve simple exponential equations, such as

(i) $3^{2x} = 27^{x+1}$

(ii) $2^{2x} + 5(2^x) - 14 = 0$